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Natural convection flow due to a heat source in a vertical channel

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Abstract

An analysis is presented of the laminar natural convection flow due to a localized heat source on the centerline of a long vertical channel or pipe whose walls are kept at a constant temperature. Stationary solutions are obtained for infinitely long and finite length channels, the asymptotic limit of infinite Rayleigh numbers is discussed, and an optimal height of the channel is found leading to maximum mass flux and minimum temperature for a given heat release rate.

1. Introduction

Natural convection in vertical channels and tubes has been studied extensively because of its interest in many practical systems, including cooling of electronic equipment [1,2], chimneys and furnaces, heat exchangers and solar energy collectors, nuclear engineering, and geophysical flows. Early experiments were carried out by Elenbaas [3] and Ostroumov [4], and later by Sparrow and Bahrami [5], for isothermal tubes and plates heated at temperatures above the ambient temperature. Bodoia and Osterle [6] analyzed the flow in a vertical channel with uniform wall temperature; Engel and Mueller [7] applied an integral method to channels of finite height with uniform wall temperature or uniform surface heat flux; Aung et al. [8,9], and later Sparrow et al. [10] and Webb and Hill [11], investigated cases with asymmetric wall conditions. A review can be seen in Gebhart et al. [12].

The "chimney" effect, or enhancement of natural convection heat transfer by confinement, has been quantified by Marsters [13] for horizontal cylinders confined by unheated vertical walls, and by Sparrow et al. [14] for horizontal finned tubes. In experiments with horizontal isothermal cylinders between vertical adiabatic plates, Karim et al. [15] found that the Nusselt number increases monotonically with the ratio of cylinder diameter to plate spacing, a result advanced by Sparrow and Pfeil [16]. Naylor and Tarasuk [17] studied numerically and experimentally the air flow induced around a vertical plate on the centerline of a channel whose walls are heated at the same temperature as the plate, obtaining values of the Nusselt number up to twice as large as for a plate in open space. Konka [18] presented flow visualizations and numerical computations for a heated horizontal cylinder in a vertical channel with its two walls kept at different temperatures.

In this paper, a related configuration is analyzed in which the vertical walls are at the ambient temperature of the fluid outside the channel and the flow is generated by a continuous supply of heat around a point on the centerline. The work is aimed at describing the laminar flow and the heat loss to the vertical walls at high values of the Rayleigh number, and at determining the mass flux in the channel as a function of its height. For the present purposes, the heat is supposed to enter directly the fluid, without an intervening solid body. In principle this would require a special device, such as focusing radiation on a region of the fluid. In practice the results that follow should be applicable when the heat is released by a body provided that its size is small compared with the width of the channel (see [19,20]) or that the drag of the body in the natural convection flow induced in the channel is small compared with the resistance of the channel walls. In either case, a separate analysis of the flow around the body is necessary to determine the Nusselt number.

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2. Formulation

Consider the planar or axisymmetric laminar natural convection flow induced by a two- or three-dimensional localized heat source of strength Q on the centerline of a long vertical channel or pipe of width or diameter $2a$ whose walls are kept at a constant temperature T_0 (ambient). A convenient temperature scale is $\Delta T = Q/ka^j$, where k is the thermal conductivity of the fluid and $j = 0,\ 1$ for planar or axisymmetric flow, respectively. In the Boussinesq approximation, the non-dimensional equations governing the flow are:

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1}$$

$$\frac{Ra}{Pr}\boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \theta \boldsymbol{i} + \nabla^2 \boldsymbol{v}, \tag{2}$$

$$Ra\, \boldsymbol{v} \cdot \nabla \theta = \nabla^2 \theta, \tag{3}$$

where $\theta = (T - T_0)/\Delta T$ is the reduced temperature, the variables $(\boldsymbol{x}, \boldsymbol{v}, p)$ are non-dimensionalized with the factors $(a, g\beta\Delta T a^2/\nu, g\beta\Delta T a)$, and $Ra = g\beta\Delta T a^3/\alpha\nu$ and $Pr = \nu/\alpha$ are the Rayleigh and Prandtl numbers. Here g is the acceleration of gravity ($\boldsymbol{i}$ is a unit vector pointing upwards) and β, ν and α are the coefficient of thermal expansion, the kinematic viscosity and the thermal diffusivity of the fluid.

In the case of an infinitely long channel, all the heat released by the source is lost to the isothermal walls. The region of warm fluid around the source has a finite length and the buoyancy force acting on the fluid in this region cannot induce a through flow in the channel. Boundary conditions appropriate for this case are:

$$y = 0 : \begin{cases} \partial u/\partial y = v = 0, \\ -y^j \partial\theta/\partial y = q(x) = \sigma^{-1} \exp[-(x/\sigma)^2], \end{cases} \tag{4}$$

$$y = 1 : u = v = \theta = 0, \tag{5}$$

$$x \to \pm\infty : u = v = \theta = 0, \tag{6}$$

where x and y are distances along the centerline and normal thereto, and u and v are the corresponding components of the velocity. Hereafter the distribution of the heat flux entering the fluid is modeled as a Gaussian of adjustable width σ. The results on the scale of the channel are insensitive to this model provided σ is small. The modification of (6) needed to account for channels of finite length will be discussed below; see (13).

3. Infinitely long channel

For the numerical treatment, Eqs. (1) and (2) were rewritten in terms of the stream function and the vorticity, then (1)–(3) were discretized using finite differences, and solved with boundary conditions (4)–(6) by means of a standard pseudotransient method. A sample numerical solution of (1)–(6) is given in Fig. 1. The flow starts as a plume immediately above the heat source, opens up to fill the channel and lose heat by conduction to the walls, and recirculates at a finite height above the source. The total length of the flow increases with Ra and depends very little on σ. Asymptotically, for large values of Ra and $Pr = O(1)$ or large, the characteristic length x_c of the region above the source where viscous forces and heat conduction extend to the whole channel cross-section, as well as the characteristic velocity u_c and temperature θ_c of the fluid in this region, can be estimated from the balance of buoyancy force and viscous force in (2): $\theta_c = u_c$; the balance of convection and lateral conduction in (3): $Ra u_c \theta_c / x_c = \theta_c$; and the global energy balance $-\int_{-\infty}^{\infty} (\partial\theta/\partial y)_1\, dx = \int_{-\infty}^{\infty} q(x)\, dx = 2\sqrt{\pi}$ (from (3)–(6)), which amounts to $\theta_c x_c = 1$. All these relations are order of magnitude balances; equality signs are used because the variables with subscript c are already characteristic values of the original variables. The three conditions above yield $x_c = Ra^{1/2}$ and $u_c = \theta_c = Ra^{-1/2}$.

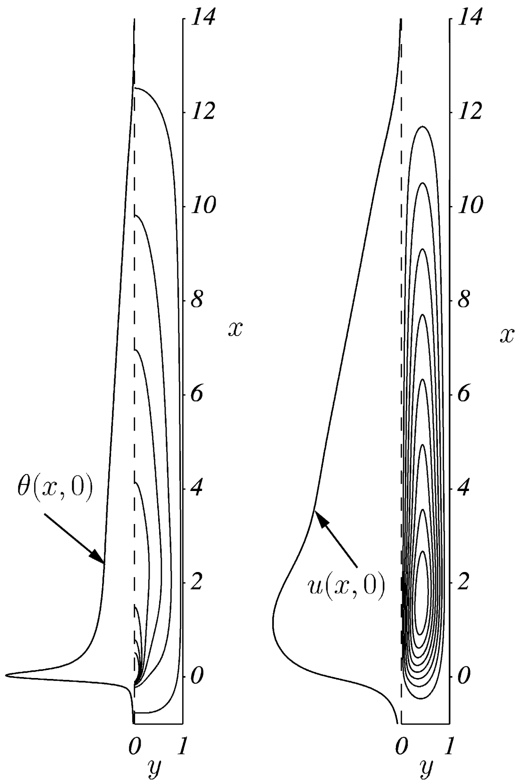

Fig. 1. Isotherms (left) and streamlines (right) for planar flow with $Ra = 10^4$, $Pr = 1$ and $\sigma = 0.1$. Also plotted are the profiles of temperature (left) and velocity (right) along the centerline.

In order to check these estimates, the vertical extent of the flow $x_{\max}$, defined as the largest value of x for which $\theta \geqslant \varepsilon\theta_{\max}$, where $\theta_{\max}$ is the maximum value of θ in the flow field and $\varepsilon = 10^{-2}$, as well as the maximum value of the stream function ψ (defined in the usual way with $\psi = 0$ on the centerline) are given in Fig. 2 as functions of the Rayleigh number. The dotted lines in this logarithmic plot have the slopes predicted by the asymptotic estimates.

Rescaling x, u and θ with x_c, u_c and θ_c, and v and p with $u_c/x_c = Ra^{-1}$ and $Ra\,u_c^2 = 1$, respectively, and then letting $Ra \to \infty$, Eqs. (2) and (3) take the boundary layer form

$$\frac{1}{Pr}\widetilde{v}\cdot\nabla\widetilde{u} = -\frac{d\widetilde{p}}{d\widetilde{x}} + \widetilde{\theta} + \nabla_T^2\widetilde{u}, \quad (7)$$

$$\widetilde{v}\cdot\nabla\widetilde{\theta} = \nabla_T^2\widetilde{\theta}, \quad (8)$$

where tildes denote rescaled variables, $\widetilde{p} = \widetilde{p}(\widetilde{x})$ only, and $\nabla_T^2 = y^{-j}\partial/\partial y(y^j\partial/\partial y)$.

The velocity and temperature profiles in Fig. 1 suggest that the flow ends at a finite $\widetilde{x} = \widetilde{x}_e(Pr)$, with the solution being locally of the form

$$\begin{aligned} &\widetilde{u} = (\widetilde{x}_e - \widetilde{x})U_1(y), \quad \widetilde{v} = V_1(y), \\ &\Delta\widetilde{p} = -\frac{1}{2}(\widetilde{x}_e - \widetilde{x})^2\Pi_1, \quad \widetilde{\theta} = (\widetilde{x}_e - \widetilde{x})\Theta_1(y). \end{aligned} \quad (9)$$

Carrying these variables into the governing equations one is led to

$$\begin{aligned} &-U_1 + y^{-j}(y^jV_1)' = 0, \\ &\frac{1}{Pr}(-U_1^2 + V_1U_1') = -\Pi_1 + \Theta_1 + \nabla_T^2U_1, \\ &-U_1\Theta_1 + V_1\Theta_1' = \nabla_T^2\Theta_1, \\ &y = 0: U_1' = V_1 = \Theta_1' = 0, \\ &y = 1: U_1 = V_1 = \Theta_1 = 0, \end{aligned} \quad (10)$$

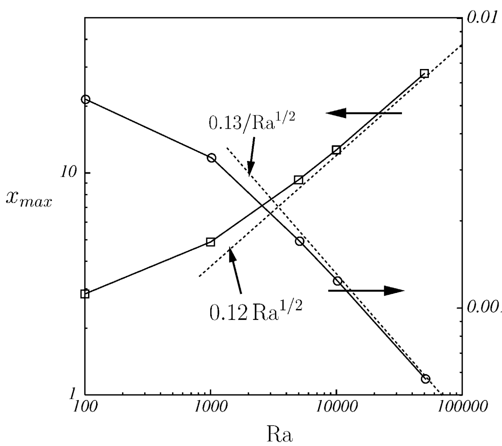

Fig. 2. Vertical extent of the flow ($x_{\max}$, squares, left-side scale) and maximum stream function ($\psi_{\max}$, circles, right-side scale) as functions of Ra for planar flow with $Pr = 1$.

where primes denote derivatives with respect to y. The solution of (10), computed with a shooting and rescaling method, is summarized in Fig. 3 in terms of the Prandtl number. The profiles of $U_1(y)$ and $\Theta_1(y)$ are given in Fig. 4 for $Pr = 1$.

Near the heat source, for $\widetilde{x} \ll 1$, the flow consists of a self-similar planar or axisymmetric laminar plume, whose solution is well known [21], and an outer recirculating flow which is cool and weak and apparently induced by the entrainment of the plume. The nature of this latter flow is different in the planar and axisymmetric cases. In the second case, which is simpler, the plume amounts to a line of sinks of uniform strength extending from $\widetilde{x} = 0$ upwards, which in the boundary layer approximation discussed above leads to an outer flow

$$\widetilde{u} = \widetilde{x}Pr\,U_0(y), \quad \widetilde{v} = Pr\,V_0(y), \quad \Delta\widetilde{p} = \frac{1}{2}\widetilde{x}^2Pr\,\Pi_0 \quad (11)$$

with (using again primes to denote y-derivatives)

$$\begin{aligned} &U_0 + y^{-j}(y^jV_0)' = 0, \\ &U_0^2 + V_0U_0' = -\Pi_0 + \nabla_T^2U_0, \\ &y = 0: V_0 \sim -\frac{\phi}{y}, \quad U_0' = 0, \\ &y = 1: U_0 = V_0 = 0, \end{aligned} \quad (12)$$

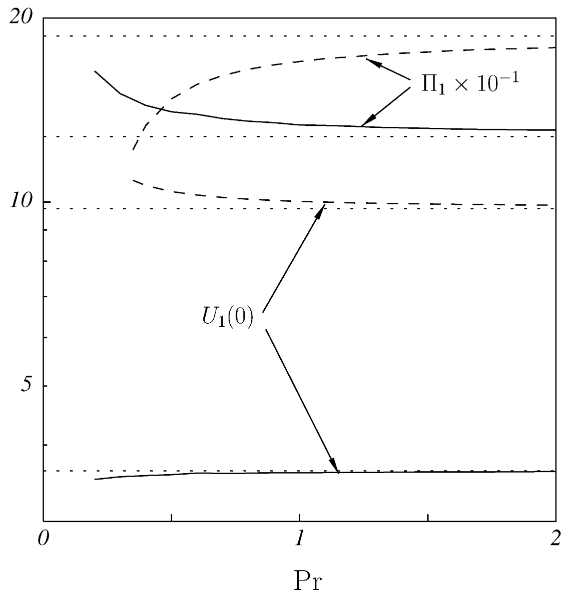

Fig. 3. Scaled pressure (Π_1) and centerline velocity [$U_1(0)$], from the solution of (10), as functions of the Prandtl number. Solid curves are for planar flow ($j = 0$) and dashed curves for axisymmetric flow ($j = 1$). Dotted lines are asymptotic values for $Pr \to \infty$.

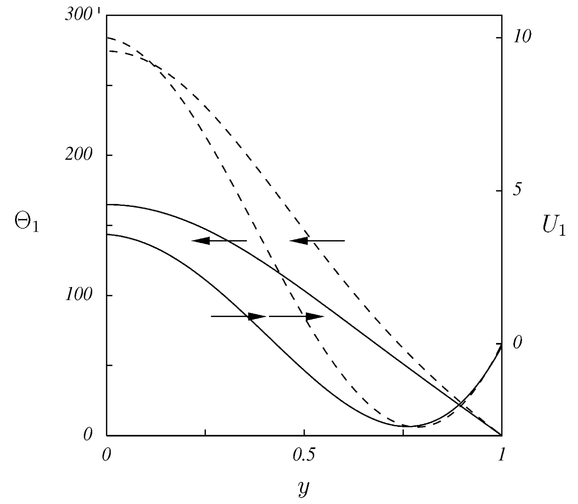

Fig. 4. Scaled temperature (left-side scale) and velocity (right-side scale) from the solution of (10) for $Pr = 1$. Solid curves are for planar flow ($j = 0$) and dashed curves for axisymmetric flow ($j = 1$).

where $2\pi\phi(Pr)$ is the entrainment rate of the plume, which is well known from the analysis of its internal structure. The solution of this problem, computed with a shooting method, is represented in Fig. 5 as a function of ϕ. The linear variation of the vertical velocity with $\widetilde{x}$ implies that convection and viscous forces remain of the same order down to $\widetilde{x} = 0$. This feature of the axisymmetric configuration was first noticed by Revuelta et al. [22] for a related problem.

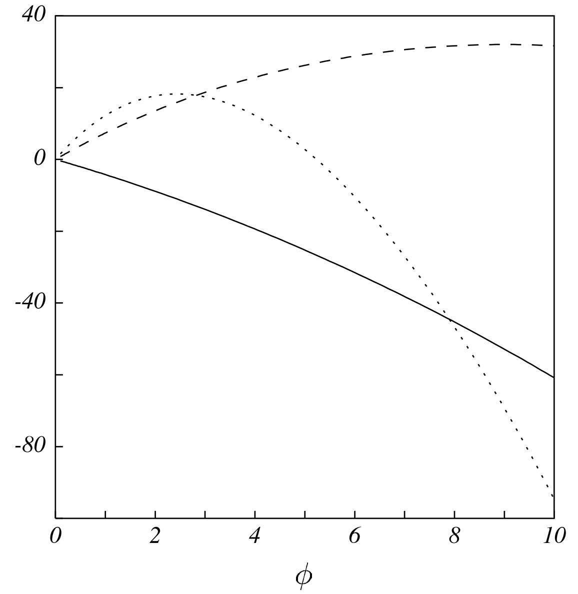

Fig. 5. Scaled centerline velocity ($U_0(0)$; solid), shear stress at the wall ($U_0'(1)$; dashed) and pressure (Π_0; dotted), from the solution of (12), as functions of ϕ.

In the planar case, the mass flux carried by the plume is proportional to $\widetilde{x}^{3/5}$, requiring a vertical downward velocity of this same order outside the plume. Then, in this outer region, convection, of $O(\widetilde{x}^{1/5})$, dominates viscous forces, of $O(\widetilde{x}^{3/5})$, when $\widetilde{x} \ll 1$, except in boundary layers of thickness proportional to $\widetilde{x}^{1/5}$ around the walls. The vorticity is of order $\widetilde{x}^{2/5}$ in these boundary layers, and should be at most of order $\widetilde{x}^{3/5}$ in the inviscid flow. The actual distributions of vorticity and temperature, however, are to be determined from the solution for $\widetilde{x} = O(1)$, where viscosity and heat conduction matter and the flow of interest occupies narrow layers by the walls. As far as the limiting Eqs. (1), (7) and (8) are concerned, the only information needed about the region $\widetilde{x} \ll 1$ is the velocity and temperature in the plume and, if some fluid recirculates without being ingested by the plume, the conditions that the temperature and the velocity are the same, on each streamline, before and after the fluid turns around.

4. Channels of finite length

Having discussed the flow in an infinitely long channel, we turn now to the case of channels of finite height. If the height H of the channel, scaled with a, is large compared with $x_c = Ra^{1/2}$, then the pressure variation induced by buoyancy in the warm fluid, $\Delta p = O(\theta_c x_c) = O(1)$, from (2), leads to a Poiseuille flow $u = U(1 - y^2)$ with $U = O(\Delta p/H) = O(u_c x_c/H) \ll u_c$ in the rest of the channel. This is because the non-dimensional adaptation length $Ra\,U$ (cf. (2)) is small compared with H when $H \ll x_c$. Though this through flow is too weak to affect the bulk of the warm region, it can still modify the conditions around the heat source and decrease the maximum temperature for a given heat release (or Ra). Thus, in the planar case, using standard estimates recast in terms of the present variables, the temperature of the plume at a distance l from the source, or the temperature of a warm body of characteristic length l, would decrease from $\theta_p = O(Ra^{-1/5} l^{-3/5})$ to $\theta_{p'} = O(Ra\,Ul)^{-1/2}$ when $l \ll l_0 = U^5 Ra^3$, and a plume with $u = O(Ra^{-3/5} l^{1/5})$ emerges and dominates the flow above this distance. On top of the warm region the velocity takes the parabolic form $u = U(1 - y^2)$ and the temperature decays exponentially: $\theta = \Theta_2 \exp(-\lambda x)$, with λ an eigenvalue. The energy Eq. (3) becomes $\nabla_T^2 \Theta_2 + \lambda U Ra(1 - y^2)\Theta_2 = 0$ at the upper reaches of the warm region, which with $\Theta_2'(0) = \Theta_2(1) = 0$ gives $\lambda U\,Ra \approx 2.828$ in the planar case and $\lambda U\,Ra \approx 7.314$ in the axisymmetric case.

The value of U increases and the maximum temperature decreases with decreasing H, until U becomes of order $u_c = Ra^{-1/2}$ when $H = O(Ra^{1/2})$. The flow in this

Table 1
Scaled inlet velocity $\widetilde{U}$ for different values of the scaled length of the channel $\widetilde{H}$; planar flow, $Pr = 1$

$\widetilde{H}$	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.50	0.60
$\widetilde{U}$	0.448	0.486	0.503	0.509	0.507	0.502	0.492	0.467	0.436

regime, in which there is no room for a Poiseuille flow to develop in the channel, is described by the boundary layer Eqs. (1), (7) and (8) and the boundary conditions (4) and (5), while (6) should be replaced by

$$\tilde{x} = -\tilde{x}_{\text{in}} : \tilde{u} = \widetilde{U}, \quad \tilde{\theta} = \tilde{p} + \frac{1}{2Pr}\widetilde{U}^2 = 0 \tag{13}$$

assuming that there are no losses of total pressure at the inlet. Here $\tilde{x}_{\text{in}}$ is the distance from the inlet of the channel to the heat source scaled with $aRa^{1/2}$ and the uniform (rescaled) inlet velocity $\widetilde{U}$ should be such that $\tilde{p} = 0$ at the outlet $\tilde{x} = \widetilde{H} - \tilde{x}_{\text{in}}$, where $\widetilde{H} = H/Ra^{1/2}$.

Finally, when H is small compared with x_c the plume does not fill the channel cross-section and most of the heat is convected away from the channel. The velocity induced around the source by the entrainment of the plume is of the order of the non-dimensional mass flux in the plume, which is $U = O(H^{3/5}/Ra^{4/5})$ in the planar case and $U = O(H/Ra)$ in the axisymmetric case. This velocity increases with H which, along with the previous estimates for tall channels, suggests that there is an optimal height $H = O(Ra^{1/2})$ leading to maximum U and minimum temperature at the heat source. This conjecture was checked by solving the boundary layer problem for $\tilde{x}_{\text{in}} \ll 1$, in which case, for planar flow

$$\tilde{\theta} = \frac{1}{(\widetilde{U}\tilde{x}_{\text{in}})^{1/2}} \exp\left(-\frac{\widetilde{U}y^2}{4\tilde{x}_{\text{in}}}\right)$$

at the inlet, immediately behind the heat source. The values of $\widetilde{U}$ computed numerically for different $\widetilde{H}$ are given in Table 1 for $Pr = 1$. A maximum $\widetilde{U} \approx 0.509$ occurs for $\widetilde{H} \approx 0.25$.

5. Conclusions

The natural convection flow induced by a localized heat source on the centerline of a vertical channel or pipe with walls at ambient temperature has been investigated numerically and asymptotically. Numerical solutions have been computed for an infinitely long channel and used to validate the asymptotic scaling for large values of a Rayleigh number based on the channel width. Simplified boundary layer equations have been written on the basis of this scaling. The vertical extent of the flow is found to be finite, and the limiting forms of the solution around the upper and lower ends have been computed. Estimates of the induced mass flux have been worked out for channels of length large and small compared with the extent of the warm region in an infinitely long channel. These estimates suggest, and numerical solution of the limiting boundary layer equations confirm, that an optimal channel length exists leading to maximum mass flux for a given heat release or Rayleigh number.

Acknowledgements

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